Entropic Repulsion for Random Interlacements

Xinyi Li, BICMR, Peking University

Joint work with Zijie Zhuang (Penn)

Nov. 25 2022

Entropic Repulsion

- In physics, an entropic force acting in a system is an emergent phenomenon resulting from the entire system's statistical tendency to increase its entropy, rather than from a particular underlying force on the atomic scale. Common example include: surface tension, elasticity of rubber, etc.

Entropic Repulsion

- In physics, an entropic force acting in a system is an emergent phenomenon resulting from the entire system's statistical tendency to increase its entropy, rather than from a particular underlying force on the atomic scale. Common example include: surface tension, elasticity of rubber, etc.
- In probability theory, this phenomenon is interpreted as follows: in many cases, a typical realization of some statistical physics model is not the configuration with the lowest energy, but rather from some collection of configurations with high cardinality (entropy).

Entropic Repulsion

- In physics, an entropic force acting in a system is an emergent phenomenon resulting from the entire system's statistical tendency to increase its entropy, rather than from a particular underlying force on the atomic scale. Common example include: surface tension, elasticity of rubber, etc.
- In probability theory, this phenomenon is interpreted as follows: in many cases, a typical realization of some statistical physics model is not the configuration with the lowest energy, but rather from some collection of configurations with high cardinality (entropy).
- The fact that BM typically travels a distance of $O(\sqrt{t})$ in time t can be regarded as a result of an entropic force.

Entropic repulsion for Gaussian free fields

- Consider a random walk bridge $W : [0, \ldots, 2N] \to \mathbb{Z}$ such that $W[0] = W[2N] = 0$. If we condition on the (rare) "hard-wall" event
$$\{W[k] \geq 0, k = 0, \ldots, 2N\},$$
then the mean value of $W[N]$ would be "pushed" to $O(\sqrt{N})$ from 0.

Entropic repulsion for Gaussian free fields

- Consider a random walk bridge $W : [0, \ldots, 2N] \to \mathbb{Z}$ such that $W[0] = W[2N] = 0$. If we condition on the (rare) “hard-wall” event
$$\{W[k] \geq 0, k = 0, \ldots, 2N\},$$
then the mean value of $W[N]$ would be “pushed” to $O(\sqrt{N})$ from 0.
- This kind of *entropic repulsion* phenomenon can also be observed in higher dimension, in the study of Gaussian free fields (as well as of other models).

Entropic repulsion for Gaussian free fields

- Consider a random walk bridge $W : [0, \ldots, 2N] \to \mathbb{Z}$ such that $W[0] = W[2N] = 0$. If we condition on the (rare) "hard-wall" event
$$\{W[k] \geq 0, k = 0, \ldots, 2N\},$$
then the mean value of $W[N]$ would be "pushed" to $O(\sqrt{N})$ from 0.
- This kind of *entropic repulsion* phenomenon can also be observed in higher dimension, in the study of Gaussian free fields (as well as of other models).
- In fact, the RW bridge itself can be regarded as a 1D Gaussian free field with Dirichlet boundary condition.

Entropic repulsion for Gaussian free fields

- Bolthausen, Deuschel and Zeitouni (*CMP*, '95) investigated the following question:

Entropic repulsion for Gaussian free fields

- Bolthausen, Deuschel and Zeitouni (*CMP*, '95) investigated the following question:
  - Consider $X : \mathbb{Z}^d \to \mathbb{R}$ the Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$. Let $K \in \mathbb{R}^d$ and K_N be its discrete blow-up.

Entropic repulsion for Gaussian free fields

- Bolthausen, Deuschel and Zeitouni (*CMP*, '95) investigated the following question:
  - Consider $X : \mathbb{Z}^d \to \mathbb{R}$ the Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$. Let $K \in \mathbb{R}^d$ and K_N be its discrete blow-up.
  - Let Ω_N stand for the event $\{X(z) \geq 0, \ \forall z \in K_N\}$. Then

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log P[\Omega_N] = c_d \mathrm{cap}_B(K)$$

  where $\mathrm{cap}_B(K)$ stands for the Newtownian capacity of K.

Entropic repulsion for Gaussian free fields

- Bolthausen, Deuschel and Zeitouni (*CMP*, '95) investigated the following question:
  - Consider $X : \mathbb{Z}^d \to \mathbb{R}$ the Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$. Let $K \in \mathbb{R}^d$ and K_N be its discrete blow-up.
  - Let Ω_N stand for the event $\{X(z) \geq 0, \ \forall z \in K_N\}$. Then

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log P[\Omega_N] = c_d \mathrm{cap}_B(K)$$

    where $\mathrm{cap}_B(K)$ stands for the Newtownian capacity of K.
  - (Deuschel and Giacomin, *CMP* '99) Moreover, as $N \to \infty$,

$$P\Big(X(\cdot) - \sqrt{c_d' \log N}\Big|\Omega_N\Big) \Rightarrow P(X(\cdot)).$$

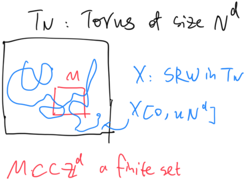
  - In other words, the field is "pushed" to infinity by the hard-wall conditioning.
- Similar results also hold for the 2D case, ($P[\Omega_N] \approx N^{-c\log N}$, field pushed to $c_2 \log N$ conditioned on Ω_N), see Bolthausen, Deuschel and Giacomin (*AoP*, '01).

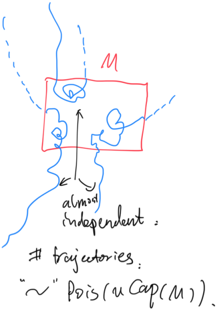
Random interlacements

- The model of random interlacements was introduced by Sznitman (*Ann. Math.* '10) to describe the local picture left by the trace of a random walk on a large discrete torus ($d \geq 3$) when it runs up to a time proportional to the volume of the torus.

Random interlacements

- The model of random interlacements was introduced by Sznitman (*Ann. Math.* '10) to describe the local picture left by the trace of a random walk on a large discrete torus ($d \geq 3$) when it runs up to a time proportional to the volume of the torus.
- It can be regarded as a Poissonian cloud of doubly-infinite trajectories, governed by an intensity parameter $u > 0$.





Random interlacemeents

- Write $\mathcal{I}^u \subset \mathbb{Z}^d$ for the union of all trajectories that appears in the cloud. We can then characterize $\mathcal{I}^u$ as a random subset of $\mathbb{Z}^d$ through the following relation

$$P[\mathcal{I}^u \cap V = \emptyset] = \exp(-u\mathrm{cap}(V))$$

where $V \subset\subset \mathbb{Z}^d$ and $\mathrm{cap}(V)$ stands for the (random walk) capacity of V.

Random interlacemeents

- Write $\mathcal{I}^u \subset \mathbb{Z}^d$ for the union of all trajectories that appears in the cloud. We can then characterize $\mathcal{I}^u$ as a random subset of $\mathbb{Z}^d$ through the following relation

$$P[\mathcal{I}^u \cap V = \emptyset] = \exp(-u\mathrm{cap}(V))$$

  where $V \subset\subset \mathbb{Z}^d$ and $\mathrm{cap}(V)$ stands for the (random walk) capacity of V.
- One can also regard random interlacements as a model of percolation with strong correlation.
- In fact,

$$\mathrm{Cov}(1_{0\in\mathcal{I}^u}, 1_{x\in\mathcal{I}^u}) \sim c(u)|x|^{2-d}.$$

- The interlacement set $\mathcal{I}^u$ is always well-connected. It is the *vacant set* $\mathcal{V}^u := \mathbb{Z}^d \setminus \mathcal{I}^u$ on which the phase transition of percolation is non-trivial.

Percolation on the vacant set

Theorem (Sznitman '10, Sidoravicius-Sznitman '08, Teixeira '08)

There exists $u_* \in (0, \infty)$ *such that*

- *for all* $u < u_*$, $\mathcal{V}^u$ *has a unique infinite cluster,* $\mathbb{P}_u$*−a.s.;*
- *for all* $u > u_*$, $\mathcal{V}^u$ *has no infinite cluster,* $\mathbb{P}_u$*−a.s.*

Percolation on the vacant set

Theorem (Sznitman '10, Sidoravicius-Sznitman '08, Teixeira '08)

There exists $u_* \in (0, \infty)$ such that

- *for all $u < u_*$, $\mathcal{V}^u$ has a unique infinite cluster, $\mathbb{P}_u$−a.s.;*
- *for all $u > u_*$, $\mathcal{V}^u$ has no infinite cluster, $\mathbb{P}_u$−a.s.*

- Let u_{**} stand for the critical parameter above which $\mathcal{V}_u$ is strongly non-percolating:

$$u_{**} := \inf\{u \geq 0; \mathbb{P}\big([-L, L]^d \overset{\mathcal{V}^u}{\leftrightarrow} \partial[-2L, 2L]^d\big) \to 0\}.$$

Percolation on the vacant set

Theorem (Sznitman '10, Sidoravicius-Sznitman '08, Teixeira '08)

There exists $u_* \in (0, \infty)$ such that

- *for all $u < u_*$, $\mathcal{V}^u$ has a unique infinite cluster, $\mathbb{P}_u-$a.s.;*
- *for all $u > u_*$, $\mathcal{V}^u$ has no infinite cluster, $\mathbb{P}_u-$a.s.*

- Let u_{**} stand for the critical parameter above which $\mathcal{V}_u$ is strongly non-percolating:

$$u_{**} := \inf\{u \geq 0; \mathbb{P}\big([-L, L]^d \overset{\mathcal{V}^u}{\leftrightarrow} \partial[-2L, 2L]^d\big) \to 0\}.$$

- One can define another critical threshold $\overline{u}$, below which $\mathcal{V}^u$ is strongly percolating.

Percolation on the vacant set

Theorem (Sznitman '10, Sidoravicius-Sznitman '08, Teixeira '08)

There exists $u_* \in (0, \infty)$ such that

- *for all $u < u_*$, $\mathcal{V}^u$ has a unique infinite cluster, $\mathbb{P}_u$−a.s.;*
- *for all $u > u_*$, $\mathcal{V}^u$ has no infinite cluster, $\mathbb{P}_u$−a.s.*

- Let u_{**} stand for the critical parameter above which $\mathcal{V}_u$ is strongly non-percolating:

$$u_{**} := \inf\{u \geq 0; \mathbb{P}\big([-L, L]^d \overset{\mathcal{V}^u}{\leftrightarrow} \partial[-2L, 2L]^d\big) \to 0\}.$$

- One can define another critical threshold $\overline{u}$, below which $\mathcal{V}^u$ is strongly percolating.
- It is trivial that $\overline{u} \leq u_* \leq u_{**}$.

Percolation on the vacant set

Theorem (Sznitman '10, Sidoravicius-Sznitman '08, Teixeira '08)

There exists $u_* \in (0, \infty)$ such that

- *for all $u < u_*$, $\mathcal{V}^u$ has a unique infinite cluster, $\mathbb{P}_u-$a.s.;*
- *for all $u > u_*$, $\mathcal{V}^u$ has no infinite cluster, $\mathbb{P}_u-$a.s.*

- Let u_{**} stand for the critical parameter above which $\mathcal{V}_u$ is strongly non-percolating:

$$u_{**} := \inf\{u \geq 0; \mathbb{P}\big([-L, L]^d \stackrel{\mathcal{V}^u}{\leftrightarrow} \partial[-2L, 2L]^d\big) \to 0\}.$$

- One can define another critical threshold $\overline{u}$, below which $\mathcal{V}^u$ is strongly percolating.
- It is trivial that $\overline{u} \leq u_* \leq u_{**}$.

A Big Open Question: Sharpness of Phase Transition

Conjecture

Do the critical parameters actually coincide, i.e.,

$$\overline{u} = u_* = u_{**}?$$

A Big Open Question: Sharpness of Phase Transition

Conjecture

Do the critical parameters actually coincide, i.e.,

$$\overline{u} = u_* = u_{**}?$$

- There is a lot of evidence that this conjecture should be true, but the strong correlation of the model hinders us from applying old and new tools to show sharpness of phase transition.

A Big Open Question: Sharpness of Phase Transition

Conjecture

Do the critical parameters actually coincide, i.e.,

$$\overline{u} = u_* = u_{**}?$$

- There is a lot of evidence that this conjecture should be true, but the strong correlation of the model hinders us from applying old and new tools to show sharpness of phase transition.
- The parallel sharpness of phase transition of the level sets of Gaussian free field has been verified recently by Duminil-Copin, Goswami, Rodriguez, Severo ('20+).

Intersection of two interlacements

- It is natural to investigate the intersection of two interlacements.

Intersection of two interlacements

- It is natural to investigate the intersection of two interlacements.
- Zhuang (*Alea* '21) investigated percolative properties.

Intersection of two interlacements

- It is natural to investigate the intersection of two interlacements.
- Zhuang (*Alea* '21) investigated percolative properties.
- It is also a nice playground for observing entropic repulsion and the so-called "Swiss cheese" picture, to be introduced later.

Intersection of two interlacements

- It is natural to investigate the intersection of two interlacements.
- Zhuang (*Alea* '21) investigated percolative properties.
- It is also a nice playground for observing entropic repulsion and the so-called "Swiss cheese" picture, to be introduced later.

Theorem (L. and Zhuang, '22+,)

(Large deviations) For a compact regular $K \subset \mathbb{R}^d$, let K_N be its discrete blow-up. Consider two independent interlacements $\mathcal{I}_1$ and $\mathcal{I}_2$ with intensities $u_1, u_2 > 0$ resp.,

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_1 \cap \mathcal{I}_2 \cap K_N = \emptyset\right] = -\frac{u_1 \wedge u_2}{d} \mathrm{cap}_B(K).$$

Intersection of two interlacements

- It is natural to investigate the intersection of two interlacements.
- Zhuang (*Alea* '21) investigated percolative properties.
- It is also a nice playground for observing entropic repulsion and the so-called "Swiss cheese" picture, to be introduced later.

Theorem (L. and Zhuang, '22+,)

(Large deviations) For a compact regular $K \subset \mathbb{R}^d$, let K_N be its discrete blow-up. Consider two independent interlacements $\mathcal{I}_1$ and $\mathcal{I}_2$ with intensities $u_1, u_2 > 0$ resp.,

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_1 \cap \mathcal{I}_2 \cap K_N = \emptyset\right] = -\frac{u_1 \wedge u_2}{d} \mathrm{cap}_B(K).$$

(Entropic repulsion) Moreover, if $u_2 < u_1$, for any $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left[\mathrm{cap}\left(K_N \cap \mathcal{I}_2\right) < \epsilon N^{d-2} \middle| \mathcal{I}_1 \cap \mathcal{I}_2 \cap K_N = \emptyset\right] = 1.$$

Comments

Theorem (L. and Zhuang, '22+)

(Large deviations) For compact regular $K \subset \mathbb{R}^d$ and let K_N be its discrete blow-up. Consider two independent random interlacements $\mathcal{I}_1$ and $\mathcal{I}_2$ with intensities $u_1, u_2 > 0$ resp.

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_1 \cap \mathcal{I}_2 \cap K_N = \emptyset\right] = -\frac{u_1 \wedge u_2}{d} \mathrm{cap}_B(K).$$

Comments

Theorem (L. and Zhuang, '22+)

(Large deviations) For compact regular $K \subset \mathbb{R}^d$ and let K_N be its discrete blow-up. Consider two independent random interlacements $\mathcal{I}_1$ and $\mathcal{I}_2$ with intensities $u_1, u_2 > 0$ resp.

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_1 \cap \mathcal{I}_2 \cap K_N = \emptyset\right] = -\frac{u_1 \wedge u_2}{d} \mathrm{cap}_B(K).$$

(Entropic repulsion) Moreover, if $u_2 < u_1$, for any $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left[\mathrm{cap}\left(K_N \cap \mathcal{I}_2\right) < \epsilon N^{d-2} \middle| \mathcal{I}_1 \cap \mathcal{I}_2 \cap K_N = \emptyset\right] = 1.$$

Comments

Theorem (L. and Zhuang, '22+)

(Large deviations) For compact regular $K \subset \mathbb{R}^d$ and let K_N be its discrete blow-up. Consider two independent random interlacements $\mathcal{I}_1$ and $\mathcal{I}_2$ with intensities $u_1, u_2 > 0$ resp.

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_1 \cap \mathcal{I}_2 \cap K_N = \emptyset\right] = -\frac{u_1 \wedge u_2}{d} \mathrm{cap}_B(K).$$

(Entropic repulsion) Moreover, if $u_2 < u_1$, for any $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left[\mathrm{cap}\left(K_N \cap \mathcal{I}_2\right) < \epsilon N^{d-2} \middle| \mathcal{I}_1 \cap \mathcal{I}_2 \cap K_N = \emptyset\right] = 1.$$

- The lower bound of the first claim is trivial.

Comments

Theorem (L. and Zhuang, '22+)

(Large deviations) For compact regular $K \subset \mathbb{R}^d$ and let K_N be its discrete blow-up. Consider two independent random interlacements $\mathcal{I}_1$ and $\mathcal{I}_2$ with intensities $u_1, u_2 > 0$ resp.

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_1 \cap \mathcal{I}_2 \cap K_N = \emptyset\right] = -\frac{u_1 \wedge u_2}{d} \mathrm{cap}_B(K).$$

(Entropic repulsion) Moreover, if $u_2 < u_1$, for any $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left[\mathrm{cap}\left(K_N \cap \mathcal{I}_2\right) < \epsilon N^{d-2} \middle| \mathcal{I}_1 \cap \mathcal{I}_2 \cap K_N = \emptyset\right] = 1.$$

- The lower bound of the first claim is trivial.
- However, the upper bound requires an involved coarse graining argument.

An Interesting Variational Problem

- In the course of the proof, we also obtain an interesting large deviation bound on the capacity of the intersection of interlacements with a macroscopic body is unusually small.

An Interesting Variational Problem

- In the course of the proof, we also obtain an interesting large deviation bound on the capacity of the intersection of interlacements with a macroscopic body is unusually small.
- Consider a regular compact $K \subset \mathbb{R}^d$ and write $\lambda_0 = \mathrm{cap}_B(K)$. For $0 < \lambda < \lambda_0$, define

$$f_K(\lambda) = \inf_{A \subset K,\ \mathrm{cap}_B(A) \leq \lambda} \mathrm{cap}_B(K \setminus A), \tag{1}$$

where the infimum runs over all "nice" subsets of K.

An Interesting Variational Problem

- In the course of the proof, we also obtain an interesting large deviation bound on the capacity of the intersection of interlacements with a macroscopic body is unusually small.
- Consider a regular compact $K \subset \mathbb{R}^d$ and write $\lambda_0 = \mathrm{cap}_B(K)$. For $0 < \lambda < \lambda_0$, define

$$f_K(\lambda) = \inf_{A \subset K,\, \mathrm{cap}_B(A) \leq \lambda} \mathrm{cap}_B(K \setminus A), \tag{1}$$

  where the infimum runs over all "nice" subsets of K.
- This variational problem seems to be very difficult. We do not know the explicit formula for f_K even when K is a cube or a ball.

An Interesting Variational Problem

- In the course of the proof, we also obtain an interesting large deviation bound on the capacity of the intersection of interlacements with a macroscopic body is unusually small.
- Consider a regular compact $K \subset \mathbb{R}^d$ and write $\lambda_0 = \mathrm{cap}_B(K)$. For $0 < \lambda < \lambda_0$, define

$$f_K(\lambda) = \inf_{A \subset K,\, \mathrm{cap}_B(A) \le \lambda} \mathrm{cap}_B(K \setminus A), \tag{1}$$

  where the infimum runs over all "nice" subsets of K.
- This variational problem seems to be very difficult. We do not know the explicit formula for f_K even when K is a cube or a ball.

Theorem (L. and Zhuang, '22+)

Given $u > 0$ and consider random interlacements $\mathcal{I}^u$, then

$$\lim_{N \to \infty} \frac{1}{N^{d-2}} \log \mathbb{P}[\mathrm{cap}\,(K_N \cap \mathcal{I}^u) < \lambda N^{d-2}] = -\frac{u}{d} f_K(d\lambda),$$

Disconnection by Random Interlacements

- Given $M > 0$ and a regular $K \subset [-M, M]^d$ and $N \in \mathbb{N}$, look at its discrete blow-up K_N. And consider the event

$$A_N \triangleq \left\{ K_N \overset{\mathcal{V}^u}{\not\longleftrightarrow} \partial S_N \right\} :$$

"no path in $\mathcal{V}^u$ connects K_N with $S_N := [-MN, MN]^d$".

Disconnection by Random Interlacements

- Given $M > 0$ and a regular $K \subset [-M, M]^d$ and $N \in \mathbb{N}$, look at its discrete blow-up K_N. And consider the event

$$A_N \triangleq \left\{ K_N \overset{\mathcal{V}^u}{\not\longleftrightarrow} \partial S_N \right\} :$$

"no path in $\mathcal{V}^u$ connects K_N with $S_N := [-MN, MN]^d$".

- When u is small such that $\mathcal{V}^u$ is (strongly) percolative, A_n is a very unlikely event. A naive bound gives $P[A_n] \geq c^{-N^{d-1}}$.

Disconnection by Random Interlacements

- Given $M > 0$ and a regular $K \subset [-M, M]^d$ and $N \in \mathbb{N}$, look at its discrete blow-up K_N. And consider the event

$$A_N \triangleq \left\{ K_N \overset{\mathcal{V}^u}{\not\longleftrightarrow} \partial S_N \right\} :$$

"no path in $\mathcal{V}^u$ connects K_N with $S_N := [-MN, MN]^d$".

- When u is small such that $\mathcal{V}^u$ is (strongly) percolative, A_n is a very unlikely event. A naive bound gives $P[A_n] \geq c^{-N^{d-1}}$.
- Question: What is the precise asymptotics of $P[A_N]$? What is the optimal strategy in realizing A_N?

Disconnection by Random Interlacements

- Given $M > 0$ and a regular $K \subset [-M, M]^d$ and $N \in \mathbb{N}$, look at its discrete blow-up K_N. And consider the event

$$A_N \triangleq \{K_N \overset{\mathcal{V}^u}{\not\longleftrightarrow} \partial S_N\} :$$

"no path in $\mathcal{V}^u$ connects K_N with $S_N := [-MN, MN]^d$".

- When u is small such that $\mathcal{V}^u$ is (strongly) percolative, A_n is a very unlikely event. A naive bound gives $P[A_n] \geq c^{-N^{d-1}}$.
- Question: What is the precise asymptotics of $P[A_N]$? What is the optimal strategy in realizing A_N?
- One can also consider the same type of disconnection by a single trajectory of random walk: let X_n be a simple random walk on $\mathbb{Z}^d$, $d \geq 3$, started from the origin and write $\mathcal{V} = \mathbb{Z}^d \setminus X[0, \infty)$. Then define A_n^0 in the same fashion as A_N, with $\mathcal{V}^u$ replaced by $\mathcal{V}$.

Disconnection by Random Interlacements and SRW

Theorem (L.-Sznitman *EJP* '14 lower bound, Sznitman *PTRF* '17 upper bound for K being a box, Nitzschner-Sznitman *JEMS* '20 upper bound for general K)

For $0 < u < u_{**}$ *and* $0 < u < \overline{u}$ *resp.,*

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log P[A_N] \geq -\frac{(\sqrt{u_{**}} - \sqrt{u})^2}{d} \mathrm{cap}_B(K);$$

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log P[A_N] \leq -\frac{(\sqrt{\overline{u}} - \sqrt{u})^2}{d} \mathrm{cap}_B(K).$$

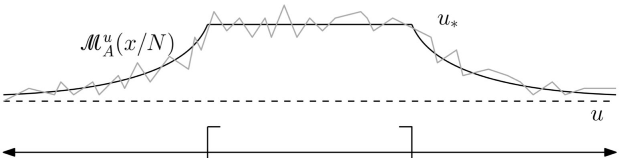
In the case of random walk, similar bounds for A_n^0 hold with $u = 0$ on the RHS' (the upper bound trivially follows from the RI case, but lower bound (L. *AoP '17*) requires substantial work and a somewhat different strategy).

Asymptotics on the Disconnection Probability

- The lower bound is obtained through the introduction of the *tilted interlacements*, i.e., RI defined on $\mathbb{Z}^d$ with inhomogeneous edge-weights which is modulated by a certain harmonic function.

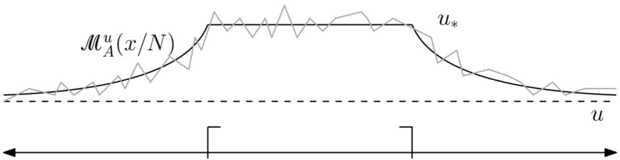
Asymptotics on the Disconnection Probability

- The lower bound is obtained through the introduction of the *tilted interlacements*, i.e., RI defined on $\mathbb{Z}^d$ with inhomogeneous edge-weights which is modulated by a certain harmonic function.



Asymptotics on the Disconnection Probability

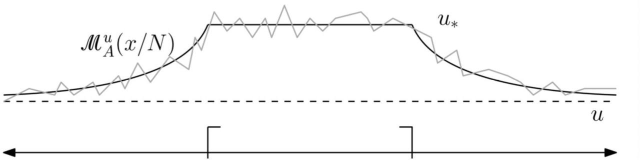
- The lower bound is obtained through the introduction of the *tilted interlacements*, i.e., RI defined on $\mathbb{Z}^d$ with inhomogeneous edge-weights which is modulated by a certain harmonic function.



- The upper bound is obtained through an extremely involved coarse graining argument and capacity estimates.

Asymptotics on the Disconnection Probability

- The lower bound is obtained through the introduction of the *tilted interlacements*, i.e., RI defined on $\mathbb{Z}^d$ with inhomogeneous edge-weights which is modulated by a certain harmonic function.



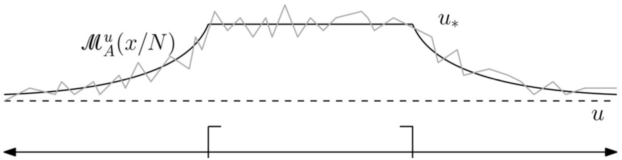
- The upper bound is obtained through an extremely involved coarse graining argument and capacity estimates.
- **If the conjecture that $\overline{u} = u_* = u_{**}$ is true**, then the upper and lower bounds coincide and we can confirm that the "tilted interlacements" indeed provides an optimal strategy for the disconnection.

Entropic Repulsion for Random Interlacements Conditioned on disconnection

- Chiarini and Nitzschner (*AoP* '20) showed that **if the conjecture**

$$\overline{u} = u_* = u_{**}$$

is indeed true, then conditioned on the disconnection event A_n, the occupation time profile of the interlacements indeed roughly coincides with that of the tilted interlacements. In other words, this suggests that conditioning on disconnection "pushes" up the intensity of interlacements from u to u_* in the bulk of K_N.



How to turn a Wiener Sausage into "Swiss Cheese"?

In 2001, van den Berg, Bolthausen, and den Hollander studied the downward (moderate) deviation of the volume of a Wiener Sausage.

How to turn a Wiener Sausage into "Swiss Cheese"?

In 2001, van den Berg, Bolthausen, and den Hollander studied the downward (moderate) deviation of the volume of a Wiener Sausage.

Theorem (BBH '01, *Ann. Math.*)

For $a > 0$, let $W^a(t)$ be the a-neighborhood of a Brownian motion in $\mathbb{R}^d$ observed until time t.

How to turn a Wiener Sausage into “Swiss Cheese”?

In 2001, van den Berg, Bolthausen, and den Hollander studied the downward (moderate) deviation of the volume of a Wiener Sausage.

Theorem (BBH '01, *Ann. Math.*)

For $a > 0$, let $W^a(t)$ be the a-neighborhood of a Brownian motion in $\mathbb{R}^d$ observed until time t. For $d \geq 3$ and $0 < b < \kappa_a$,

$$\lim_{t\to\infty} \frac{1}{t^{(d-2)/d}} \log P\left(|W^a(t)| \leq bt\right) = I_d(a,b),$$

where $I_d(a,b)$ is a rate function with a certain variational representation.

How to turn a Wiener Sausage into "Swiss Cheese"?

In 2001, van den Berg, Bolthausen, and den Hollander studied the downward (moderate) deviation of the volume of a Wiener Sausage.

Theorem (BBH '01, *Ann. Math.*)

For $a > 0$, let $W^a(t)$ be the a-neighborhood of a Brownian motion in $\mathbb{R}^d$ observed until time t. For $d \geq 3$ and $0 < b < \kappa_a$,

$$\lim_{t\to\infty} \frac{1}{t^{(d-2)/d}} \log P\left(|W^a(t)| \leq bt\right) = I_d(a,b),$$

where $I_d(a,b)$ is a rate function with a certain variational representation.

They also show that the optimal strategy to realize the above large deviation is for W to "form a Swiss cheese": this Wiener sausage covers part of the space, leaving random holes of $O(1)$ size whose density varies on scale $t^{1/d}$ according to a certain optimal profile.

Wiener Sausage into "Swiss Cheese", cont.

In 2004, the same authors studied the (upward) large deviation of the volume of the intersection of two Wiener sausages.

Wiener Sausage into "Swiss Cheese", cont.

In 2004, the same authors studied the (upward) large deviation of the volume of the intersection of two Wiener sausages.

Theorem (BBH '04, *Ann. Math.*)

For $a > 0$*, let* $W_1^a(t)$ *and* $W_2^a(t)$ *be the a-neighborhoods of two independent standard Brownian motions in* $\mathbb{R}^d$ *starting at* 0 *and observed until time* t*.*

Wiener Sausage into "Swiss Cheese", cont.

In 2004, the same authors studied the (upward) large deviation of the volume of the intersection of two Wiener sausages.

Theorem (BBH '04, *Ann. Math.*)

For $a > 0$, let $W_1^a(t)$ and $W_2^a(t)$ be the a-neighborhoods of two independent standard Brownian motions in $\mathbb{R}^d$ starting at 0 and observed until time t. For $d \geq 3$ and $c > 0$,

$$\lim_{t\to\infty} \frac{1}{t^{(d-2)/d}} \log P\left(|W_1^a(ct) \cap W_2^a(ct)| \geq t\right) = I_d(a, c),$$

where $I_d(a, c)$ is a rate function with a certain variational representation.

Wiener Sausage into "Swiss Cheese", cont.

In 2004, the same authors studied the (upward) large deviation of the volume of the intersection of two Wiener sausages.

Theorem (BBH '04, *Ann. Math.*)

For $a > 0$, let $W_1^a(t)$ and $W_2^a(t)$ be the a-neighborhoods of two independent standard Brownian motions in $\mathbb{R}^d$ starting at 0 and observed until time t. For $d \geq 3$ and $c > 0$,

$$\lim_{t\to\infty} \frac{1}{t^{(d-2)/d}} \log P\left(|W_1^a(ct) \cap W_2^a(ct)| \geq t\right) = I_d(a, c),$$

where $I_d(a, c)$ is a rate function with a certain variational representation.

Again, the optimal strategy to realize the above large deviation is for W_1 and W_2 independently to "form a Swiss cheese": they cover part of the space, leaving random holes of $O(1)$ size whose density varies on scale $t^{1/d}$ according to a certain optimal profile.

Swiss Cheese in the discrete

- Phetpradap (Thesis '12) obtained the respective results in the discrete.

Swiss Cheese in the discrete

- Phetpradap (Thesis '12) obtained the respective results in the discrete.
- L. (AoP '17) applied the *tilted interlacements* as the (plausible) optimal strategy in computing the asymptotic probability in the disconnection-by-simple-random-walk problem.

Swiss Cheese in the discrete

- Phetpradap (Thesis '12) obtained the respective results in the discrete.
- L. (AoP '17) applied the *tilted interlacements* as the (plausible) optimal strategy in computing the asymptotic probability in the disconnection-by-simple-random-walk problem.
- Sznitman '17 suggested that "the *tilted interlacements* possibly offer in a discrete context a microscopic model for the type of 'Swiss cheese' picture advocated in [BBH01]".

Swiss Cheese in the discrete

- Phetpradap (Thesis '12) obtained the respective results in the discrete.
- L. (AoP '17) applied the *tilted interlacements* as the (plausible) optimal strategy in computing the asymptotic probability in the disconnection-by-simple-random-walk problem.
- Sznitman '17 suggested that "the *tilted interlacements* possibly offer in a discrete context a microscopic model for the type of 'Swiss cheese' picture advocated in [BBH01]".
- Asselah and Schapira ('17, '20) recently have some partial progress towards understanding the nature of "Swiss cheese" in the discrete.

Swiss Cheese in the discrete

- Phetpradap (Thesis '12) obtained the respective results in the discrete.
- L. (AoP '17) applied the *tilted interlacements* as the (plausible) optimal strategy in computing the asymptotic probability in the disconnection-by-simple-random-walk problem.
- Sznitman '17 suggested that "the *tilted interlacements* possibly offer in a discrete context a microscopic model for the type of 'Swiss cheese' picture advocated in [BBH01]".
- Asselah and Schapira ('17, '20) recently have some partial progress towards understanding the nature of "Swiss cheese" in the discrete.
- For the problem we investigated, we conjecture that tilted interlacements also enters into play in the conditioning.

Swiss Cheese in the discrete

- Phetpradap (Thesis '12) obtained the respective results in the discrete.
- L. (AoP '17) applied the *tilted interlacements* as the (plausible) optimal strategy in computing the asymptotic probability in the disconnection-by-simple-random-walk problem.
- Sznitman '17 suggested that "the *tilted interlacements* possibly offer in a discrete context a microscopic model for the type of 'Swiss cheese' picture advocated in [BBH01]".
- Asselah and Schapira ('17, '20) recently have some partial progress towards understanding the nature of "Swiss cheese" in the discrete.
- For the problem we investigated, we conjecture that tilted interlacements also enters into play in the conditioning.
- Conditional on $(\mathcal{I}_1 \cap \mathcal{I}_2) \cap K_N = \emptyset$, when N tends to ∞, we expect that the law of $(\mathcal{I}_1, \mathcal{I}_2)$ should resemble more and more $(\mathcal{I}^{u_1}, \widetilde{\mathcal{I}})$ where $\widetilde{\mathcal{I}}$ represents a kind of tilted interlacements with intensity 0 in K_N and u_2 at infinity, indendent of $\mathcal{I}^{u_1}$.

Thanks for your attention!